Lecture 10

Wednesday, February 5, 2020 5:33 AM

- Proof of Lemma 1, Cor 1-2 in Lecture 9 notes.

Recall: Thm 1. Let $\Omega \subseteq \mathbb{C}^n$. TFAE:

(i) Ω is a d.o. holom.

(ii) $\forall K \subset\subset \Omega$, $\hat{K}_\Omega \subset\subset \Omega$.

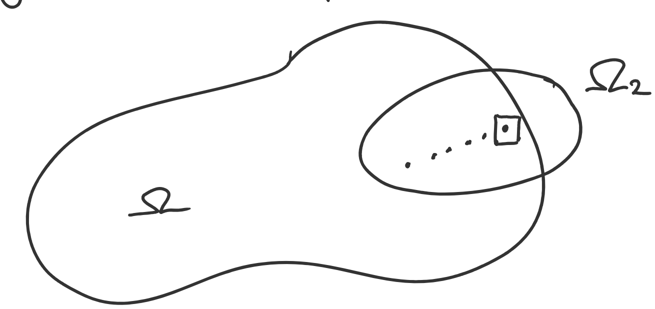
(iii) $\exists f \in \mathcal{O}(\Omega)$ that does not extend across any bdry pt. I.e. $\nexists\, \Omega_2 \not\subseteq \Omega$, $\emptyset \neq \Omega_1 \subseteq \Omega \cap \Omega_2$, $F \in \mathcal{O}(\Omega_2)$ s.t. $F|_{\Omega_1} = f$.

Pf. (i)$\Rightarrow$(ii) is Cor. 2 from Lecture 9.

(iii)$\Rightarrow$(i) is immediate from def. of d.o. holom.

(ii)$\Rightarrow$(iii). WLOG assume Ω connected. Let $D^n = \mathbb{D}^n = \{z \in \mathbb{C}^n : |z_j| < 1\}$, $\Delta(z) = \Delta_{\Omega, D^n}(z) = \sup\{r > 0 : \{z\} + rD^n \subseteq \Omega\}$ as in Lemma 1, and let $D_z = \{z\} + \Delta(z)\,\mathbb{D}^n \subseteq \Omega$. Thus, D_z is largest polydisk of "shape" $\mathbb{D}^n$ that is centered at z and contained in Ω.

Let M be a countable dense subset of Ω. Suffices to construct $f \in \mathcal{O}(\Omega)$ s.t. f does not extend to open neighborhood of $\overline{D_\zeta}$ for any $\zeta \in M$. For, if Ω_2, Ω_1 as in (iii) exist then $\exists \zeta \in M$ s.t. $\overline{D_\zeta} \subseteq \Omega_2$:



Now let $\{\zeta_j\}_{j=1}^\infty$ be a sequence in Ω s.t. each ζ in M appears ∞ many times. Let $\{K_j\}_{j=1}^\infty$ be an exhaustion of Ω by compact sets ($K_1 \subseteq K_2 \subseteq \dots$, $\Omega = \bigcup_{j=1}^\infty K_j$, and $\forall K \subset\subset \Omega$, $K \subseteq K_j$, $j \gg 1$).

Since $\hat{K}_j = \widehat{(K_j)}_\Omega \subset\subset \Omega$ by assumption and $\overline{D_{\zeta_j}} \cap \partial\Omega \neq \emptyset$

Since $\hat{K}_j = \widehat{(K_j)}_\Omega \subset\subset \Omega$ by assumption and $\overline{D_{\zeta_j}} \cap \partial\Omega \neq \emptyset$ by construction, $\exists\, z_j \in D_{\zeta_j} \setminus \hat{K}_j$. Hence, $\exists\, f_j \in \mathcal{O}(\Omega)$ s.t. $f_j(z_j) = 1$ and $\sup_{K_j} |f_j| < 1$. By replacing f_j by f_j^p, $p \gg 1$, WLOG assume $f_j(z_j) = 1$, $\sup_{K_j} |f_j| < \frac{1}{2^j}$. Consider the ∞ product:

$$f = \prod_{j=1}^{\infty} (1 - f_j)^j. \qquad (1)$$

We know (Math 220B or C) that this product converges to holom. function in Ω s.t. $f(z) = 0$ only when $f_j(z) = 1$, some j, $\iff \forall K \subset\subset \Omega$, $\sum_{j=1}^{\infty} j \sup_K |f_j| < \infty$

But $K \subseteq K_j$, $j \geq N$. Since $\sup_{K_j} |f_j| < \frac{1}{2^j}$ and $\sum_{j=1}^{\infty} j \frac{1}{2^j} < \infty$,

it follows that (1) converges a holom. fcn f s.t. $f \not\equiv 0$ and $f_{z^\alpha}(z_j) = 0$, $\forall\, |\alpha| < j$. Pick $\zeta \in M$, $\exists$ subsequence $\zeta_{j_\ell} = \zeta$ and hence $z_{j_\ell} \in D_\zeta \setminus \hat{K}_j$. Going to a subseq. if necessary WLOG assume $z_{j_\ell} \to z_0 \in \partial D_\zeta$. Since f vanishes to order at least $j_\ell - 1$ at z_{j_ℓ}, if f extended as holom. function in nbhd of $\overline{D_\zeta}$, we would have $f_{z^\alpha}(z_0) = 0$, $\forall \alpha \Rightarrow f \equiv 0$ in D_ζ and hence in Ω. Since $f \not\equiv 0$ by construction, the pf is complete. ☒